Growth dynamics of noise-sustained structures in nonlinear optical resonators

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Abstract: The existence of macroscopic noise-sustained structures in nonlinear optics is theoretically predicted and numerically observed, in the regime of convective instability. The advection-like term, necessary to turn the instability to convective for the parameter region where advection overwhelms the growth, can stem from pump beam tilting or birefringence induced walk-off. The growth dynamics of both noise-sustained and deterministic patterns is exemplified by means of movies. This allows to observe the process of formation of these structures and to confirm the analytical predictions. The amplification of quantum noise by several orders of magnitude is predicted. The qualitative analysis of the near- and far-field is given. It suffices to distinguish noise-sustained from deterministic structures; quantitative informations can be obtained in terms of the statistical properties of the spectra.

OCIS codes: (190.4420) Nonlinear optics, transverse effects in; (270.2500) Fluctuations, relaxations, and noise

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1. Introduction

Optical patterns offer the very attractive possibility of studying the interface between classical and quantum patterns. Macroscopic and spatially structured manifestations of quantum correlations are foreseen to occur in these patterns [1]. Such correlations are expected since, at a microscopic level, the physical mechanism behind the pattern formation process is often the simultaneous emission of twin photons, four wave mixing processes or other processes involving highly correlated photons. Correlations are easily observed in the far field and should encode specific features of quantum statistics.
In the search for the manifestations of quantum noise in optical patterns it is natural to look for situations in which noise is enhanced or amplified. Critical fluctuations close to an instability point are one of these situations recently considered [2, 3,4]. The noisy precursor, observed just below threshold anticipates the pattern to be formed beyond the instability. This precursor occurs because fluctuations with the wavenumber to be selected above threshold become weakly damped as the threshold is approached.
A more dramatic manifestation of noise occurs above a convective instability threshold [5]. Here fluctuations are amplified (instead of being weakly damped) while being advected away from the system. This gives rise to macroscopic structures that are continuously regenerated by noise and hence the name of noise sustained patterns. This phenomenon acts as a microscope, with amplification factors of several orders of magnitude, to observe noise and its spatially dependent manifestations [6]. In this paper we give two examples of noise sustained optical patterns. Our emphasis here is in showing how these patterns grow dynamically from noise, invading part of the system and being there maintained by noise.
The two examples to be considered are paradigmatic in the field of optical pattern formation and quantum noise properties. The first is a cavity filled by a Kerr type nonlinear medium and pumped by an external laser beam [6]. This system was a prototype model for pattern formation in optics [7,8] and it has also been where the question of quantum fluctuations in patterns was first addressed [9]. The second example is an optical parametric oscillator (OPO), also a paradigm for studies of pattern formation [10] and generation of squeezed and nonclassical light [11]. A necessary condition for the existence of a convective instability is the presence of an advection-like term in the governing equations; this term can have different origins. In our first example this originates in any pump misalignement; we will study this example in a simple transverse one-dimensional geometry to clarify the main concepts. For the OPO we consider a type-I phase matching in a uniaxial crystal. Here, the advection term originates in the walk-off between the ordinary and extraordinary rays, due to birefringence.

Thus, the outline of the paper is as follows. In Section 2 we briefly recall the definition of the convectively unstable regime and the linear stability analysis which allows to determine the different regimes. In Section 3, we describe the convective instabilities and noise sustained structures in Kerr nonlinear resonators with one transverse dimension. We show in two movies the growth dynamics of the pattern in the convectively and absolutely unstable regimes. The role of the noise in sustaining the structure in the convective regime is clearly exemplified. The distinction between these two regimes manifests qualitatively in the time evolution of the far field of the pattern. Section 4 is devoted to noise-sustained structures in OPO with two transverse dimensions. We show the diagram in parameter space where the zero solution becomes unstable either convectively or absolutely. Two movies, displaying the growth dynamics of the noise-sustained and the deterministic patterns, are also presented, for the near- and the far-field. Finally we compare a snapshot of the pattern formed in each of the two cases with the noisy precursor below threshold. Conclusions are presented in section V.

2. Definitions and linear stability analysis

We start by briefly recalling the notion of convective and absolute instabilities; readers may refer to [5,6,12] for more details. The steady-state of a generic system is defined to be absolutely stable (unstable) when a perturbation decays (grows) with time. However, a third possibility is that the perturbation grows (i.e. is unstable) but at the same time is advected so quickly that, at a fixed position, it actually decays. In this case the state is called convectively unstable. Note that the definition is unambiguous only if a fixed frame of reference is defined; in the cases we consider here the fixed frame corresponds to the pump beam.

The usual linear stability analysis of the steady-state can be re-formulated in order to take into account the above distinction of convectively and absolutely unstable regimes. In general, the calculation of the pump amplitude thresholds of the instability for the systems we are considering entails the evaluation of the linearized asymptotic behaviour of a generic perturbation of the steady-state [5,12]. The convective threshold turns out to correspond to the instability threshold which can be calculated as if the advection was not present, i.e. $\Re[\lambda(\vec{q})] > 0$ where $\lambda(\vec{q})$ is the linearized eigenvalue of largest real part and $\vec{q}$ the wave-vector of the perturbation. This means that at threshold all unstable modes are convectively unstable. As regards the threshold of the absolutely unstable regime, its determination reduces to the calculation of the pump amplitude which satisfies the following conditions:

$$\begin{aligned} \Re[\lambda(\vec{k}_s)] &> 0 \\ \Re[\nabla^2_{\vec{k}}\lambda(\vec{k}=\vec{k}_s)] &\geq 0 \end{aligned} \tag{1}$$

where the complex vector $\vec{k}_s$ defined by

$$\nabla_{\vec{k}}\lambda(\vec{k}=\vec{k}_s) = 0 \tag{2}$$

is a saddle point for $\Re[\lambda(\vec{k}_s)]$ in the complex vector space.

A detailed mathematical explication of the procedure to get the above conditions can be found in [5,6,12]. Here, for the sake of simplicity we just mention that wave-vectors $\vec{q}$ are extended to the complex space $\vec{k}$ in order to evaluate the integral, in the wave-vector space, which determines the asymptotic linearized evolution of a perturbation.

3. Kerr resonators

For a nonlinear resonator containing a Kerr medium [7] an advection-like term can stem from the input pump beam tilting [13]. A one-dimensional (1D), transversal model

is used in order to simplify the analysis and to clarify the main concepts. The 1D assumption can be also justified from an experimental viewpoint [14].
The equation governing the electric field $A(x,t)$ is [6,7,13]:

$$\partial_t A - 2\alpha_0 \partial_x A = i\partial_x^2 A - [1 + i\eta(\Delta - |A|^2)]A + E_0 + \sqrt{\epsilon}\xi(x,t)\,, \tag{3}$$

where: α_0 represents the tilt angle, η the sign of the nonlinearity, Δ the cavity detuning and E_0 the pump. Diffraction is represented by the first term on the r.h.s. and mirror losses by the first in the squared brackets (exact definitions can be found in [6]). We have introduce a complex additive noise $\xi(x,t)$, Gaussian, with zero-mean and correlation $\langle \xi(x,t), \xi^*(x',t')\rangle = 2\delta(x-x')\delta(t-t')$, which is a standard semiclassical model of noise. For the linearized version of the Langevin equations of the optical parametric oscillator a similar term describes quantum noise in the Wigner representation, as considered in [2]. In our case it can also account for thermal and input field fluctuations.
Through conditions (1,2) applied to the linearized eigenvalue of eq. (3) we have estimated the threshold of the absolute instability for a fixed set of parameters for the steady-state A_0, solution of $A_0[1 + i\eta(\Delta - |A_0|^2)] = E_0$. For the same parameters we have integrated the equation for a pump amplitude slightly above this threshold and slightly below, in the regime of convective instability.
The time evolution of the near-field (A) and far-field (the space Fourier transform of A) of eq. (3) confirm that different, unstable regimes actually exist. In movie 1 the near field intensity time evolution (left side) can be observed, for the pump amplitude above the threshold of absolute instability.

Movie 1. Near field (left) and far field (right) growth dynamics in the absolutely unstable regime.

The initial condition is the steady-state plus a weak perturbation: noise is not applied ($\epsilon = 0$) because it is not necessary to generate the pattern. After a certain transitory a drifting structure is generated. In spite of the drift, the pattern tends to invade all the flat top region where the pump is above the threshold. The evolution of the far-field (right) shows that after the transitory, well defined harmonics are generated (due to the multiple wave mixing). Their linewidth is scarcely influenced by the presence of noise as demonstrated in [6] in the equivalent time analysis.

Movie 2. Near field (left) and far field (right) growth dynamics in the convectively unstable regime.

In the convective regime (movie 2) we applied noise ($\epsilon \simeq 10^{-5}$) and a pattern forms again. However, note that the structure, even for long times does not invade all the system but rather gathers its saturated value at a random spatial position. This is due to the fact that noise needs to drift for enough time in order to be amplified. When the noise source is turned off the pattern (after the delay due to the drifting) eventually disappears. In the far-field, the noticable broadening of the spectral lines with respect to the previous case confirms the different, noisy, nature of the pattern observed.

4. Parametric oscillators

In the optical parametric oscillator, i.e. when the nonlinear medium inside the resonator has a quadratic response, the advection-like term stems naturally from the birefringence of the nonlinear crystal, which is exploited to phase-match the nonlinear interaction. In fact, in a birefringent medium the ordinary and extraordinary polarizations can be subject to a transversal walk-off [15]. In particular, we consider, a degenerate, type I OPO (scattered photons are thus frequency and polarization degenerate). The pump

($A_0(x,y,t)$ at frequency $2\omega_0$) and the signal ($A_1(x,y,t)$ at frequency ω_0) evolution is described by the following set of coupled equations [10,16]:

$$
\begin{aligned}
\partial_t A_0 &= \gamma_0[-(1+i\Delta_0)A_0 + E_0 + ia_0\nabla^2 A_0 + 2iK_0A_1^2] + \sqrt{\epsilon_0}\xi_0(x,y,t) \\
\partial_t A_1 &= \gamma_1[-(1+i\Delta_1)A_1 + \rho_1\partial_y A_1 + ia_1\nabla^2 A_1 + iK_0A_1^* A_0] + \sqrt{\epsilon_1}\xi_1(x,y,t) \quad (4)
\end{aligned}
$$

where: $\gamma_{0,1}$ represent the losses, $\Delta_{0,1}$ the detunings, $a_{0,1}$ the diffraction, K_0 the nonlinearity, ρ_1 the walk-off, E_0 the pump (see [10,12] for details). Noise terms $\xi_{0,1}$ have the same characteristics of the Kerr case and are uncorrelated.
The uniform steady-state, whose stability we are interested in, is: $A_0 = E_0/(1+i\Delta_0)$, $A_1 = 0$. It turns out that it can become unstable along the signal component of the eigenvector, A_1, and thus it is necessary to consider only one linearized equation. We can calculate the predicted absolute instability thresholds through conditions (1,2) with λ determined from eqs. (4). In summary, the stability diagram for the OPO is presented in figure 1 as a function of the signal detuning Δ_1.

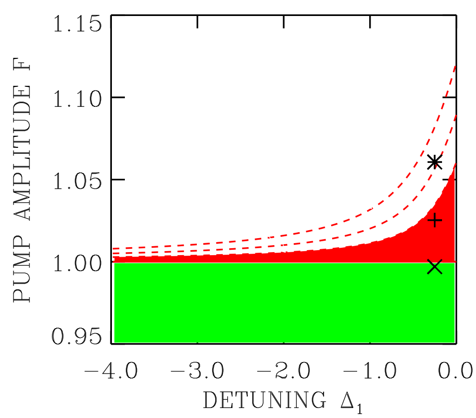

Fig. 1. Stability diagram for the OPO. Shadowed regions are: stable (green), convectively unstable (red). The white region indicates the absolute instability. Absolute threshold shifts upwards by increasing α_0 (red dashed curves).

The linear analysis also reveals that the first mode to become unstable satisfies $q_x = 0$, i.e. is parallel to the x axis. This stems from the breaking of the rotational system due to the walk-off. The walk-off does not affect the growth rate but rather the spreading velocity of the perturbation. The first mode to become absolutely unstable is that which balances the advection with spreading [12].
The growth dynamics for A_1 is shown in movie 3, for the absolutely unstable, and in movie 4 for the convectively unstable regime. The pump A_0 was a supergaussian beam and we show in the movie the central region of space where the pump was flat.

Movie 3. Near field (left) and far field (right) growth dynamics in the absolutely unstable regime.

In the initial stage, noise generates a randomly oriented pattern in both cases; later the two evolutions start to differ. In the former case the stripes generated are parallel to the x-axis, as predicted. The deterministic pattern invades the whole region of pumping, stripes are well defined and no defects of the horizontal orientation can be observed waiting a long enough time (see also figure 2, top, which corresponds to the snapshot of the final time of the evolution).

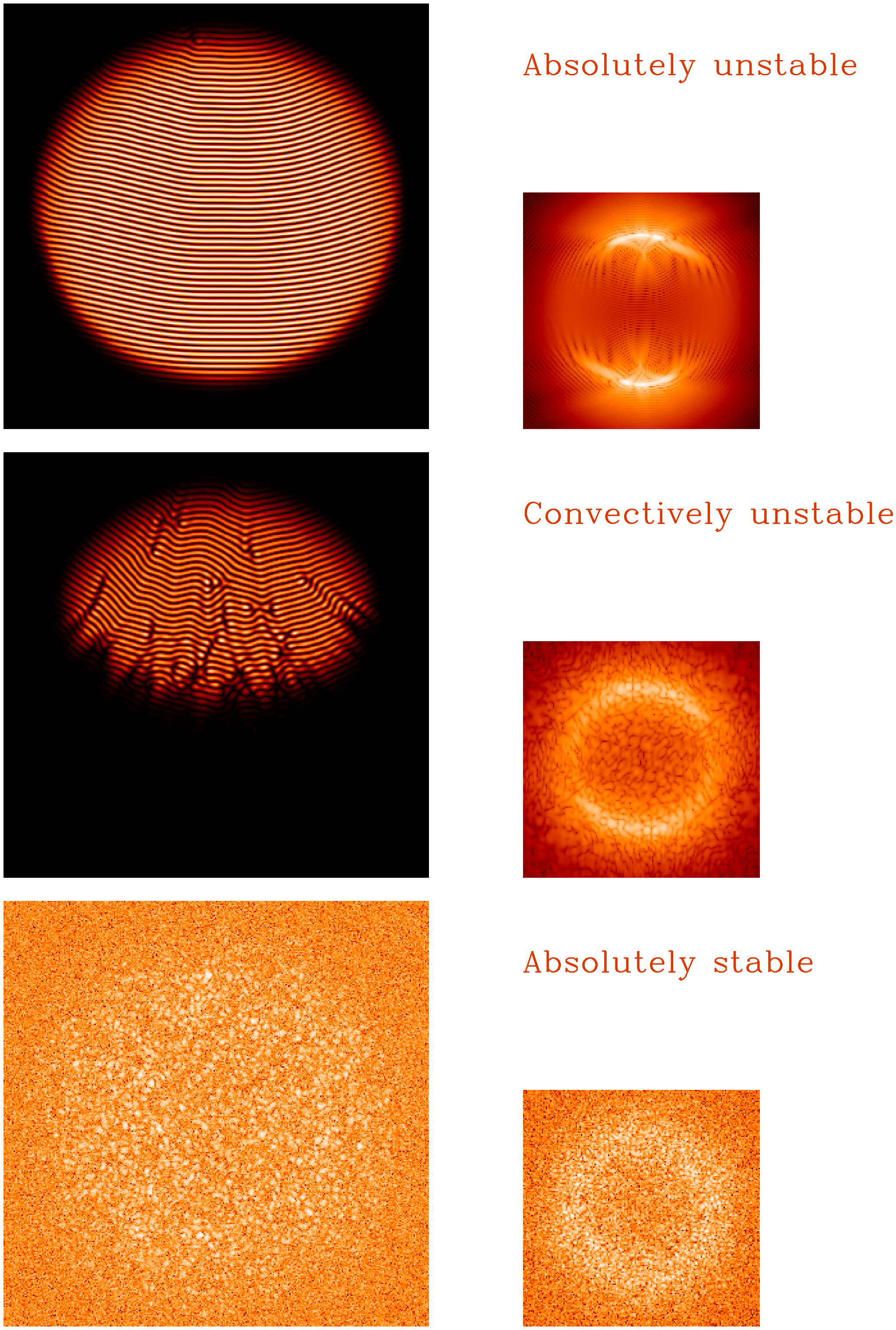

Fig. 2. Snapshots of the near(far)-field at time $t = 2000$ on the left (right) hand side. Parameteres of the top, middle and bottom images correspond respectively to (*, +, X) of Fig. 1.

In the convective regime (movie 4) the pattern is continuously generated by noise with random orientation at the bottom of the window and amplified while drifting. During this process stripes get parallel to the x-axis. Note that the dynamical orientation is less marked than in the previous case and defects can be still seen. The location where the pattern gathers its saturated value is random as in the Kerr case (see figure 2, middle). The average space delay and its variance depends on the noise intensity.

Movie 4. Near field (left) and far field (right) growth dynamics in the convectively unstable regime.

The far field observation suffices to distinguish the two different regimes. At the first stage all modes on a ring of radius $q_c = \sqrt{-\Delta_1/a_1}$ are excited; later, in the absolute regime two narrow spots form, in correspondance with the first mode that become absolutely unstable ($q_x = 0$), in the convective regime two broadenend arcs of the ring remain visible even for very large times (see figure 2b).
Quantitative results which help to sharply distinguish the two regimes can be obtained by means of a time spectral analysis [12]. To summarize we present three situations in figure 2, i.e. from the top to the bottom: absolutely unstable, convectively unstable and absolutely stable (close to threshold). The first is a deterministic pattern, the second a noise-sustained one and the last is a noisy-precursor we have referred to in the introduction. Signatures of a deterministic pattern are: the high intensity, the pattern orientation (if 2D), orthogonal to the drift direction due to the symmetry breaking, the fact that it invades all the system, narrow spatial dispersion in the far field. Noise-sustained structures show: high intensity due to large noise magnification factors, preferential selection of the stripe orientation (in 2D), although defects are clearly observable, only partial and random occupancy of the system, broadened far fields. Noisy precursors below the instability threshold are characterized by: low intensities and random orientation (in 2D), because noise is only selectively enhanced by the filtering effect of the nonlinearity, and very broaden far field.

5. Conclusions

We have theoretically predicted the existence of macroscopic, noise-sustained transversal structures in nonlinear optical resonators. Numerical solutions confirm the qualitative and quantitative predictions. Noise-sustained structures can be found in the regime of convective instability which can be induced either by a tilt in the input pump beam or by the walk-off due to birefringence. The growth dynamics of noise-sustained as well as deterministic patterns is presented and helps to distinguish the nature of the structures.

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